Jacobs Journal of Computer Science

Review Article

Strong Insertion of a Contra-α−Continuous Function

Majid Mirmiran[1*] and Binesh Naderi[2]

[1]Department of Mathematics, University of Isfahan, Isfahan 81746-73441, Iran

[2]Department of General Courses, School of Management and Medical Information Sciences, Isfahan University of Medical Sciences, Isfahan, Iran

*Corresponding author: Majid Mirmiran, Department of Mathematics, University of Isfahan, Isfahan 81746-73441, Iran; E-mail: mirmir@sci.ui.ac.ir

Received Date: 05-02-2019

Accepted Date: 05-08-2019

Published Date: 05-12-2019

Abstract

Necessary and sufficient conditions in terms of lower cut sets are given for the insertion of a contra-α−continuous function between two comparable real-valued functions.

Keywords: Insertion; Strong binary relation; Semi-open set; Preopen set; α−open set; Lower cut set

Introduction

The concept of a preopen set in a topological space was introduced by H.H. Corson and E. Michael in 1964 [1-4]. A subset A of a topological space (X,τ) is called *preopen* or *locally dense* or *nearly open* if $A \subseteq Int(Cl(A))$. A set A is called *pre-closed* if its complement is preopen or equivalently if $Cl(Int(A)) \subseteq A$. The term, preopen, was used for the first time by A.S. Mashhour, M.E. Abd El-Monsef and S.N. El-Deeb [5-20], while the concept of a, locally dense, set was introduced by H.H. Corson and E. Michael [4]. The concept of a semi-open set in a topological space was introduced by N. Levine in 1963 [17]. A subset A of a topological space (X,τ) is called *semiopen* [10] if $A \subseteq Cl(Int(A))$. A set A is called *semi-closed* if its complement is semi-open or equivalently if $Int(Cl(A)) \subseteq A$.

Recall that a subset A of a topological space (X,τ) is called α−*open* if A is the difference of an open and a nowhere dense subset of X. A set A is called α−*closed* if its complement is α−open or equivalently if A is union of a closed and a nowhere dense set. We have a set is α−open if and only if it is semi-open and preopen. A generalized class of closed sets was considered by Maki in [19]. He investigated the sets that can be represented as union of closed sets and called them V-sets. Complements of V-sets, i.e., sets that are intersection of open sets are called Λ-sets [19-23].

Recall that a real-valued function f defined on a topological space X is called A-continuous [24] if the preimage of every open subset of R belongs to A, where A is a collection of subsets of X. Most of the definitions of function used throughout this paper are consequences of the definition of A-continuity. However, for unknown concepts the reader may refer to [5,11].

In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity.

J. Dontchev in [6] introduced a new class of mappings called contra continuity. S. Jafari and T. Noiri in [12,13] exhibited and studied among others a new weaker form of this class of mappings called contra-α−continuous. A good number of researchers have also initiated different types of contra continuous like mappings in the papers [1, 3, 8,9,10 and 23].

Hence, a real-valued function f defined on a topological space X is called *contra-α−continuous* (resp. *contra-semi-continuous*, *contra-pre-continuous*) if the pre-image of every open subset of R is α−closed (resp. *semi*-closed, pre-closed) in X [6].

Results of Katˇetov [14, 15] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give a necessary and sufficient conditions for the insertion of a contra-α−continuous function between two comparable real valued functions.

If g and f are real-valued functions defined on a space X, we write $g \leq f$ (resp. $g<f$) in case $g(x) \leq f(x)$ (resp. $g(x)<f(x)$) for all x in X.

The following definitions are modifications of conditions considered in [16].

A property P defined relative to a real-valued function on a topological space is a *cα−property* provided that any constant function has property P and provided that the sum of a function with property P and any contraα−continuous function also has property P. If $P1$ and $P2$ are *cα*−property, the following terminology is used:(i) A space X has the *weak cα−insertion property* for $(P1,P2)$ if and only if for any functions g and f on X such that $g \leq f$, g has property $P1$ and f has property $P2$, then there exists a contraα−continuous function h such that $g \leq h \leq f$.(ii) A space X has the *strong cα−insertion property* for $(P1,P2)$ if and only if for any functions g and f on X such that $g \leq f$, g has property $P1$ and f has property $P2$, then there exists a contra-α−continuous function h such that $g \leq h \leq f$ and if $g(x)<f(x)$ for any x in X, then $g(x)<h(x)<f(x)$.

In this paper, for a topological space whose α−kernel of sets are α−open, is given a sufficient condition for the weak cα−insertion property. Also, for a space with the weak cα−insertion property, we give necessary and sufficient conditions for the space to have the strong cα−insertion property. Several insertion theorems are obtained as corollaries of these results.

The Main Result

Before giving a sufficient condition for insert ability of a contra-α−continuous function, the necessary definitions and terminology are stated. The abbreviations *cαc*, *cpc* and *csc* are used for contra-α−continuous, contra-pre-continuous and contra-*semi*-continuous, respectively.

Let (X,τ) be a topological space, the family of all α−open, α−closed, semi-open, semi-closed, preopen and preclosed will be denoted by $\alpha O(X,\tau)$, $\alpha C(X,\tau)$, $sO(X,\tau)$, $sC(X,\tau)$, $pO(X,\tau)$ and $pC(X,\tau)$, respectively.

Definition 2.1: Let A be a subset of a topological space (X,τ). We define the subsets A^{Λ} and A^{V} as follows:
$A^{\Lambda}=\cap\{O : O \supseteq A, O \in (X,\tau)\}$ and $A^{V}=\cup\{F : F \subseteq A, F^c \in (X,\tau)\}$.
In [7, 18, 22], A^{Λ} is called the *kernel* of A.

We define the subsets $\alpha(A^{\Lambda})$,$\alpha(A^{V})$,$p(A^{\Lambda})$,$p(A^{V})$,$s(A^{\Lambda})$ and $s(A^{V})$ as follows:
$\alpha(A^{\Lambda})=\cap\{O : O \supseteq A, O \in \alpha O(X,\tau)\}$ $\alpha(A^{V}) = \cup\{F : F \subseteq A, F \in \alpha C(X,\tau)\}$, $p(A^{\Lambda}) = \cap\{O : O \supseteq A, O \in pO(X,\tau)\}$, $p(A^{V})=\cup\{F : F \subseteq A, F \in pC(X,\tau)\}$, $s(A^{\Lambda})=\cap\{O : O \supseteq A, O \in sO(X,\tau)\}$ and $s(A^{V})=\cup\{F : F \subseteq A, F \in sC(X,\tau)\}$.
$\alpha(A^{\Lambda})$ (resp. $p(A^{\Lambda})$, $s(A^{\Lambda})$) is called the α − *kernel* (resp. *pre-kernel*, *semi-kernel*) of A.

The following first two definitions are modifications of conditions considered in [14, 15].

Definition 2.2: If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows: $x \bar{\rho} y$ if and only if $y \rho v$ implies $x \rho v$ and $u \rho x$ implies $u \rho y$ for any u and v in S.

Definition 2.3: A binary relation ρ in the power set $P(X)$ of a topological space X is called a *strong binary relation* in $P(X)$ in case ρ satisfies each of the following conditions:

1. If $A_i \rho B_j$ for any $i \in\{1,...,m\}$ and for any $j \in\{1,...,n\}$, then there exists a set C in $P(X)$ such that $A_i \rho C$ and $C \rho B_j$ for any $i \in\{1,...,m\}$ and any $j \in\{1,...,n\}$.

2. If $A \subseteq B$, then $A \bar{\rho} B$.

3. If $A \rho B$, then $\alpha(A^{\wedge}) \subseteq B$ and $A \subseteq \alpha(B^{V})$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:

Definition 2.4: If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < \ell\} \subseteq A(f,\ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number ℓ, then $A(f,\ell)$ is called a *lower indefinite cut set* in the domain of f at the level ℓ.

We now give the following main result:

Theorem 2.1: Let g and f be real-valued functions on the topological space X, in which α−kernel sets are α−open, with $g \leq f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t1<t2$ then $A(f,t1)\ \rho\ A(g,t2)$, then there exists a contra-α−continuous function h defined on X such that $g \leq h \leq f$. **Proof.** Let g and f be real-valued functions defined on the X such that $g \leq f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t1<t2$ then $A(f,t1)\ \rho\ A(g,t2)$.

Define functions F and G mapping the rational numbers Q into the power set of X by $F(t)=A(f,t)$ and $G(t)=A(g,t)$. If $t1$ and $t2$ are any elements of Q with $t1<t2$, then $F(t1)\ \bar{\rho}\ F(t2), G(t1)\ \bar{\rho}\ G(t2)$, and $F(t1)\ \rho\ G(t2)$. By Lemmas 1 and 2 of [15] it follows that there exists a function H mapping Q into the power set of X such that if $t1$ and $t2$ are any rational numbers with $t1<t2$, then $F(t1)\ \rho\ H(t2), H(t1)\ \rho\ H(t2)$ and $H(t1)\ \rho\ G(t2)$.
For any x in X, let $h(x)=\inf\{t \in Q : x \in H(t)\}$.

We first verify that $g \leq h \leq f$: If x is in $H(t)$ then x is in $G(t')$ for any $t'>t$; since x is in $G(t')=A(g, t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence $g \leq h$. If x is not in $H(t)$, then x is not in $F(t')$ for any $t'<t$; since x is not in $F(t')=A(f,t')$ implies that $f(x)>t'$, it follows that $f(x) \geq t$. Hence $h \leq f$. Also, for any rational numbers $t1$ and $t2$ with $t1<t2$, we have $h^{-1}(t1,t2)=\alpha(H(t2)^{V}) \setminus \alpha(H(t1)^{\wedge})$. Hence $h^{-1}(t1,t2)$ is α−closed in X, i.e., h is a contra *alpha*−continuous function on X. The above proof used the technique of theorem 1 in [14].

Theorem 2.2: Let $P1$ and $P2$ be $c\alpha$−property and X be a space that satisfies the weak $c\alpha$−insertion property for ($P1$, $P2$). Also assume that g and f are functions on X such that $g \leq f$, g has property $P1$ and f has property $P2$. The space X has the strong $c\alpha$−insertion property for ($P1$,$P2$) if and only if there exist lower cut sets $A(f-g, 2^{-n})$ and there exists a sequence $\{H_n\}$ of subsets of X such that (i) for each n, H_n and $A(f-g, 2^{-n})$ are completely separated by contra-α−continuous functions, and ($\{x \in X : (f-g)(x) > 0\} = \bigcup_{n=1}^{\infty} H_n$ ii) . **Proof.** Theorem 3.1, of [21].

Theorem 2.3: Let $P1$ and $P2$ be $c\alpha$−properties and assume that the space X satisfied the weak $c\alpha$−insertion property for ($P1$,$P2$). The space X satisfies the strong $c\alpha$−insertion property for ($P1$, $P2$) if and only if X satisfies the strong $c\alpha$−insertion property for ($P1$,$c\alpha c$) and for ($c\alpha c$,$P2$). **Proof.** Theorem 3.2, of [21].

Application

Before stating the consequences of theorems 2.1, 2.2 and 2.3 we suppose that X is a topological space whose α−kernel sets are α−open.

Corollary 3.1: If for each pair of disjoint preopen (resp. *semi*−open) sets $G1$,$G2$ of X, there exist α−closed sets $F1$ and $F2$ of X such that $G1 \subseteq F1$, $G2 \subseteq F2$ and $F1 \cap F2 = \emptyset$ then X has the weak $c\alpha$−insertion property for
(cpc,cpc) (resp. (csc,csc)).

Proof. Let g and f be real-valued functions defined on X, such that f and g are cpc (resp. csc), and $g \leq f$. If a binary relation ρ is defined by $A \rho B$ in case $p(A^{\wedge}) \subseteq p(B^{V})$ (resp. $s(A^{\wedge}) \subseteq s(B^{V})$), then by hypothesis ρ is a strong binary relation in the power set of X. If $t1$ and $t2$ are any elements of Q with $t1<t2$, then

$A(f,t1) \subseteq \{x \in X : f(x) \leq t1\} \subseteq \{x \in X : g(x) < t2\} \subseteq A(g,t2)$;

since $\{x \in X : f(x) \leq t1\}$ is a preopen (resp. *semi*-open) set and since $\{x \in X : g(x)<t2\}$ is a preclosed (resp. *semi*-closed) set, it follows that $p(A(f,t1)^{\wedge}) \subseteq p(A(g,t2)^{V})$ (resp. $s(A(f,t1)^{\wedge}) \subseteq s(A(g,t2)^{V})$). Hence $t1<t2$ implies that $A(f,t1)\ \rho\ A(g,t2)$. The proof follows from Theorem 2.1.

Corollary 3.2: If for each pair of disjoint preopen (resp. *semi*−open) sets $G1$,$G2$, there exist α−closed sets $F1$ and $F2$ such that $G1 \subseteq F1$, $G2 \subseteq F2$ and $F1 \cap F2 = \emptyset$ then every contra-precontinuous (resp. contra-*semi*−continuous) function is contra-α−continuous.

Proof. Let f be a real-valued contra-pre-continuous (resp. contra-*semi*−continuous) function defined on X. Set $g=f$, then by Corollary 3.1, there exists a contraα−continuous function h such that $g=h=f$.

Corollary 3.3: If for each pair of disjoint preopen (resp. *semi*−open) sets $G1,G2$ of X , there exist α−closed sets $F1$ and $F2$ of X such that $G1 \subseteq F1$, $G2 \subseteq F2$ and $F1 \cap F2=\emptyset$ then X has the strong $c\alpha$−insertion property for (cpc,cpc) (resp. (csc,csc)).

Proof. Let g and f be real-valued functions defined on the X, such that f and g are cpc (resp. csc), and $g \le f$. Set $h=(f+g)/2$, thus $g \le h \le f$ and if $g(x) < f(x)$ for any x in X, then $g(x)<h(x)<f(x)$. Also, by Corollary 3.2, since g and f are contra-α−continuous functions hence h is a contraα−continuous function.

Corollary 3.4: If for each pair of disjoint subsets $G1,G2$ of X , such that $G1$ is preopen and $G2$ is *semi*−open, there exist α−closed subsets $F1$ and $F2$ of X such that $G1 \subseteq F1$, $G2 \subseteq F2$ and $F1 \cap F2 = \emptyset$ then X have the weak $c\alpha$−insertion property for (cpc,csc) and (csc,cpc).

Proof. Let g and f be real-valued functions defined on X, such that g is cpc (resp. csc) and f is csc (resp. cpc), with $g \le f$.If a binary relation ρ is defined by $A\ \rho\ B$ in case $s(A^{\Lambda}) \subseteq p(B^{V})$ (resp. $p(A^{\Lambda}) \subseteq s(B^{V})$), then by hypothesis ρ is a strong binary relation in the power set of X. If $t1$ and $t2$ are any elements of Q with $t1<t2$, then

$A(f,t1) \subseteq \{x \in X : f(x) \le t1\} \subseteq \{x \in X : g(x) < t2\} \subseteq A(g,t2)$;

since $\{x \in X : f(x) \le t1\}$ is a *semi*−open (resp. preopen) set and since $\{x \in X : g(x) < t2\}$ is a preclosed (resp. *semi*−closed) set, it follows that $s(A(f,t1)^{\Lambda}) \subseteq p(A(g,t2)^{V})$ (resp. $p(A(f,t1)^{\Lambda}) \subseteq s(A(g,t2)^{V})$). Hence $t1 < t2$ implies that $A(f,t1)\ \rho\ A(g,t2)$. The proof follows from Theorem 2.1.

Before stating consequences of Theorem 2.2 and 2.3 we state and prove the necessary lemmas.

Lemma 3.1: The following conditions on the space X are equivalent:

1. For each pair of disjoint subsets $G1,G2$ of X, such that $G1$ is preopen and $G2$ is *semi*−open, there exist α−closed subsets $F1,F2$ of X such that $G1 \subseteq F1,G2 \subseteq F2$ and $F1 \cap F2=\emptyset$.

2. If G is a *semi*−open (resp. preopen) subset of X which is contained in a preclosed (resp. *semi*−closed) subset F of X, then there exists an α−closed subset H of X such that $G \subseteq H \subseteq \alpha(H^{\Lambda}) \subseteq F$.

Proof. (i) ⇒ (ii) Suppose that $G \subseteq F$, where G and F are *semi*−open (resp. preopen) and preclosed (resp. *semi*−closed) subsets of X, respectively. Hence, F^c is a preopen (resp. *semi*−open) and $G \cap F^c = \emptyset$.

By (i) there exists two disjoint α−closed subsets $F1,F2$ such that $G \subseteq F1$ and $F^c \subseteq F2$. But

$$F^c \subseteq F_2 \Rightarrow F_2^c \subseteq F$$

and hence

$$F_1 \cap F_2 = \varnothing \Rightarrow F_1 \subseteq F_2^c$$

$$G \subseteq F_1 \subseteq F_2^c \subseteq F$$

And since F_2^c is an α−open subset containing $F1$, we conclude that $\alpha(F_1^{\Lambda}) \subseteq F_2^c$, i.e.,

$$G \subseteq F_1 \subseteq \alpha(F_1^{\Lambda}) \subseteq F.$$

By setting $H = F1$, condition (ii) holds.

(ii) ⇒ (i) Suppose that $G1,G2$ are two disjoint subsets of X, such that $G1$ is preopen and $G2$ is *semi*−open.

This implies that $G_2 \subseteq G_1^c$ and G_1^c is a preclosed subset of X. Hence by (ii) there exists an α−closed set H such that

$$G_2 \subseteq H \subseteq \alpha(H^{\Lambda}) \subseteq G_1^c.$$

But

$$H \subseteq \alpha(H^{\Lambda}) \Rightarrow H \cap \alpha((H^{\Lambda})^c) = \emptyset$$

and

$$\alpha(H^{\Lambda}) \subseteq G_1^c \Rightarrow G_1 \subseteq \alpha((H^{\Lambda})^c)$$

Furthermore, $\alpha((H^{\Lambda})^c)$ is an α−closed subset of X. Hence $G2 \subseteq H,G1 \subseteq \alpha((H^{\Lambda})^c)$ and $H \cap \alpha((H^{\Lambda})^c)=\emptyset$. This means that condition (i) holds.

Lemma 3.2: Suppose that X is a topological space. If each pair of disjoint subsets $G1,G2$ of X, where $G1$ is preopen and $G2$ is *semi*−open, can be separated by α−closed subsets of X then there exists a contra-α−continuous function $h : X \to [0,1]$ such that $h(G2)=\{0\}$ and $h(G1)=\{1\}$.

Proof. Suppose $G1$ and $G2$ are two disjoint subsets of X, where $G1$ is preopen and $G2$ is *semi*−open. Since $G1 \cap G2=\emptyset$, hence $G_2 \subseteq G_1^c$. In particular, since G_1^c is a pre-closed subset of X containing the *semi*−open subset $G2$ of X,by Lemma 3.1, there exists an α−closed subset $H_{1/2}$ such that

$$G_2 \subseteq H_{1/2} \subseteq \alpha(H_{1/2}^{\Lambda}) \subseteq G_1^c.$$

Note that $H_{1/2}$ is also a pre-closed subset of X and contains $G2$, and G_1^c is a pre-closed subset of X and contains the *semi*−open subset $\alpha(H^{\Lambda}_{1/2})$ of X.

Hence, by Lemma 3.1, there exists α−closed subsets $H_{1/4}$ and $H_{3/4}$ such that

$G_2 \subseteq H_{1/4} \subseteq \alpha(H^{\Lambda}_{1/4}) \subseteq H_{1/2} \subseteq \alpha(H^{\Lambda}_{1/2}) \subseteq H_{3/4} \subseteq \alpha$

. By continuing this method for every $t \in D$, where $D \subseteq [0,1]$ is the set of rational numbers that their denominators are exponents of 2, we obtain α−closed subsets H_t with the property that if $t1, t2 \in D$ and $t1<t2$, then $H_t1 \subseteq H_t2$. We define the function h on X by $h(x)$=inf {t: $x \in H_t$} for x 6∈ $G1$ and $h(x)$=1 for $x \in G1$.

Note that for every $x \in X$, $0 \le h(x) \le 1$, i.e., h maps X into [0,1]. Also, we note that for any $t \in D$, $G2 \subseteq H_t$; hence $h(G2)$={0}. Furthermore, by definition, $h(G1)$={1}. It remains only to prove that h is a contra-α−continuous function on X. For every α ∈ R, we have if α ≤ 0 then {$x \in X$: $h(x)<\alpha$}=∅ and if 0<α then {$x \in X$: $h(x)<\alpha$}=∪{H_t: $t<\alpha$}, hence, they are α−closed subsets of X. Similarly, if α<0 then {$x \in X$: $h(x)>\alpha$}=X and if 0 ≤ α then $\{x \in X : h(x) > \alpha\} = \cup\{\alpha((H^{\Lambda}_t)^c) : t > \alpha\}$

hence, every of them is an α−closed subset. Consequently, h is a contra-α−continuous function.

Lemma 3.3: Suppose that X is a topological space. If each pair of disjoint subsets $G1$,$G2$ of X, where $G1$ is preopen and $G2$ is *semi*−open, can separate by α−closed subsets of X, and $G1$ (resp. $G2$) is an α−closed subsets of X, then there exists a contra-continuous function $h : X \to [0,1]$ such that, $h^{-1}(0) = G1$ (resp. $h^{-1}(0)=G2$) and $h(G2)$={1} (resp. $h(G1)$={1}). **Proof.** Suppose that $G1$ (resp. $G2$) is an α−closed subset of X. By Lemma 3.2, there exists a contra-α−continuous function h: $X \to [0, 1]$ such that, $h(G1)$={0} (resp. $h(G2)$={0}) and $h(X \setminus G1)$={1} (resp. $h(X \setminus G2)$={1}). Hence, $h^{-1}(0)=G1$ (resp. $h^{-1}(0)=G2$) and since $G2 \subseteq X\setminus G1$ (resp. $G1 \subseteq X\setminus G2$), therefore $h(G2)$={1} (resp. $h(G1)$={1}).

Lemma 3.4: Suppose that X is a topological space such that every two disjoint *semi*−open and preopen subsets of X can be separated by α−closed subsets of X. The following conditions are equivalent:

1. For every two disjoint subsets $G1$ and $G2$ of X, where $G1$ is preopen and $G2$ is *semi*−open, there exists a contra-α−continuous function $h : X \to [0,1]$ such that, $h^{-1}(0) = G1$ (resp. $h^{-1}(0) = G2$) and $h^{-1}(1) = G2$ (resp. $h^{-1}(1) = G1$).
2. Every preopen (resp. *semi*−open) subset of X is a α−closed subset of X.
3. Every pre-closed (resp. *semi*−closed) subset of X is a α−open subset of X.

Proof: (i) ⇒ (ii) Suppose that G is a preopen (resp. *semi*−open) subset of X. Since ∅ is a *semi*−open (resp. preopen) subset of X, by (i) there exists a contra-α−continuous function $h: X \to [0, 1]$ such that, $h^{-1}(0)=G$. Set $F_n = \{x \in X : h(x) < \frac{1}{n}\}$ .Then for every $n \in$N, F_n is an α−closed subset of X and $\bigcap_{n=1}^{\infty} F_n = \{x \in X : h(x) = 0\} = G$.

(ii) ⇒ (i) Suppose that $G1$ and $G2$ are two disjoint subsets of X, where $G1$ is preopen and $G2$ is *semi*−open. By Lemma 3.3, there exists a contraα−continuous function f: $X \to [0, 1]$ such that, $f^{-1}(0)=G1$ and $f(G2)$={1}. Set $G = \{x \in X : f(x) < \frac{1}{2}\}$, $F = \{x \in X : f(x$,and $H = \{x \in X : f(x) > \frac{1}{2}\}$. Then $G \cup F$ and $H \cup F$ are two α−open subsets of X and $(G\cup F) \cap G2=\emptyset$. By Lemma 3.3, there exists a contra-α−continuous function $g : X \to [\frac{1}{2},1]$ such that, $g^{-1}(1)=G2$ and $g(G \cup F) = \{\frac{1}{2}\}$. Define h by $h(x) = f(x)$ for $x \in G\cup F$, and $h(x)=g(x)$ for $x \in H\cup F$. Then h is well defined and a contra-α−continuous function, since $(G\cup F)\cap(H\cup F)=F$ and for every $x \in F$ we have $f(x) = g(x) = \frac{1}{2}$. Furthermore, $(G\cup F) \cup (H \cup F)=X$, hence h defined on X and maps to [0,1]. Also, we have $h^{-1}(0) = G1$ and $h^{-1}(1) = G2$.

(ii) ⇔ (iii) By De Morgan law and noting that the complement of every α−open subset of X is an α−closed subset of X and complement of every α−closed subset of X is an α−open subset of X, the equivalence is hold.

Corollary 3.5: If for every two disjoint subsets $G1$ and $G2$ of X, where $G1$ is preopen (resp. *semi*−open) and $G2$ is *semi*−open (resp. preopen), there exists a contra-α−continuous function $h: X \to [0, 1]$ such that, $h^{-1}(0)=G1$ and $h^{-1}(1)=G2$ then X has the strong *cα*−insertion property for (*cpc,csc*) (resp. (*csc,cpc*)).

Conclusion and Proof

Since for every two disjoint subsets $G1$ and $G2$ of X, where $G1$ is preopen (resp. *semi*−open) and $G2$ is *semi*−open (resp. preopen), there exists a contra-α−continuous function $h : X \to [0,1]$ such that, $h^{-1}(0)=G1$ and $h^{-1}(1)=G2$, define $F_1 = \{x \in X : h(x) < \frac{1}{2}\}$ and $F_2 = \{x \in X : h(x) > \frac{1}{2}\}$. Then $F1$ and $F2$ are two disjoint α−closed subsets of X

that contain $G1$ and $G2$, respectively. Hence by Corollary 3.4, X has the weak $c\alpha$–insertion property for (*cpc,csc*) and (*csc,cpc*). Now, assume that g and f are functions on X such that $g \leq f$, g is *cpc* (resp. *csc*) and f is $c\alpha c$. Since $f–g$ is *cpc* (resp. *csc*), therefore the lower cut set $A(f - g, 2^{-n})=\{x \in X : (f - g)(x) \leq 2^{-n}\}$ is a preopen (resp. *semi*–open) subset of X. Now setting $H_n=\{x \in X : (f - g)(x)>2^{-n}\}$ for every $n \in \mathbb{N}$, then by Lemma 3.4, H_n is an α–open subset of X and we have $\{x \in X : (f-g)(x) > 0\} = \bigcup_{n=1}^{\infty} H_n$ and for every $n \in \mathbb{N}$, H_n and $A(f - g, 2^{-n})$ are disjoint subsets of X. By Lemma 3.2, H_n and $A(f - g, 2^{-n})$ can be completely separated by contra-α–continuous functions. Hence by Theorem 2.2, X has the strong $c\alpha$–insertion property for (*cpc*,$c\alpha$c) (resp. (*csc*,$c\alpha$c)).

By an analogous argument, we can prove that X has the strong $c\alpha$–insertion property for ($c\alpha c$,*csc*) (resp. ($c\alpha c$,*cpc*)). Hence, by Theorem 2.3, X has the strong $c\alpha$–insertion property for (*cpc,csc*) (resp. (*csc,cpc*)).

(i)

Acknowledgement

This research was partially supported by Centre of Excellence for Mathematics (University of Isfahan). MSC (2010): Primary 54C08, 54C10, 54C50; Secondary 26A15, 54C30.

References

1. Al-Omari A and Md Noorani MS. Some properties of contra-b-continuous and almost contra-b-continuous functions. European J Pure Appl Math 2009;2(2): 213-230.
2. Brooks F. Indefinite cut sets for real functions. Amer Math Monthly 1971; 78:1007-1010.
3. Caldas M and Jafari S. Some properties of contra-β–continuous functions. Mem Fac Sci Kochi Univ 2001; 22:19-28.
4. Corson HH and Michael E. Metrizability of certain countable unions, Illinois J Math 1964; 8: 351-360.
5. Dontchev. The characterization of some peculiar topological space via α– and β–sets. Acta Math Hungar 1995; 69(1-2): 67-71.
6. Dontchev J. Contra-continuous functions and strongly S-closed space. Intrnat J Math Math Sci 1996; 19(2): 303-310.
7. Dontchev J and Maki H. On sg-closed sets and semi–λ–closed sets. Questions Answers Gen Topol 1997; 15(2): 259-266.
8. Ekici E. On contra-continuity. Annales Univ Sci Bodapest 2004; 47: 127-137.
9. Ekici E. New forms of contra-continuity. Carpathian J Math 2008;24(1): 37-45.
10. El-Magbrabi AI. Some properties of contra-continuous mappings. Int J General Topol 2010; 3(1-2): 55-64.
11. Ganster MI. A decomposition of continuity. Acta Math Hungar 1990; 56(3-4): 299-301.
12. Jafari S and Noiri T. Contra-continuous function between topological spaces. Iranian Int J Sci 2001; 2: 153-167.
13. Jafari S and Noiri T. On contra-precontinuous functions. Bull Malaysian Math Sc Soc 2002; 25: 115-128.
14. Katetov M. On real-valued functions in topological spaces. Fund Math 1951; 38: 85-91.
15. Katetov M. Correction to on real-valued functions in topological spaces. Fund Math 1953; 40: 203-205.
16. Lane E. Insertion of a continuous function. Pacific J Math 1976;66: 181-190.
17. Levine N. Semi-open sets and semi-continuity in topological space. Amer Math Monthly 1963;70: 36-41.
18. Maheshwari SN and Prasad R. On R_{Os}–spaces. Portugal Math 1975; 34:213-217.
19. Maki H. Generalized Λ–sets and the associated closure operator, The special Issue in commemoration of Prof. Kazuada IKEDA's Retirement1986; 139-146.
20. Mashhour AS, Abd El-Monsef ME and El-Deeb SN. On precontinuous and weak pre-continuous mappings. Proc Math Phys Soc Egypt 1982; 53: 47-53.
21. Mirmiran M. Insertion of a function belonging to a certain subclass of R^X. Bull Iran Math Soc 2002; 28(2): 19-27.
22. Mrsevic M. On pairwise R and pairwise $R1$ bitopological spaces. Bull Math Soc Sci Math RS Roumanie 1986; 30: 141-145.
23. Nasef AA. Some properties of contra-continuous functions. Chaos Solitons Fractals 2005; 24: 471-477.
24. Przemski M. A decomposition of continuity and α–continuity. Acta Math Hungar 1993; 61(1-2): 93-98.